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STANDARD CLASSES ON THE BLOW-UP OF $\mathbb{P}^n$ AT POINTS IN VERY GENERAL POSITION

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ABSTRACT. We study conjectures on the dimension of linear systems on the blow-up of $\mathbb{P}^2$ and $\mathbb{P}^3$ at points in very general position. We provide algorithms and Maple code based on these conjectures.

INTRODUCTION

In this note we consider classes $D \in \mathrm{Pic}(X)$, where X is the blow-up of $\mathbb{P}^n$ at r points in very general position. We recall that the dimensions of the cohomology groups of any line bundle whose class is D do not depend on the choice of the representative. We will denote these dimensions with $h^i(D)$. Given $D := dH - \sum_i m_i E_i$, where H is the pull-back of the class of a hyperplane and the E_i's are the classes of exceptional divisors, it is not hard to see (Proposition 1.2) that if $d > 0$ and $m_i \geq 0$, then $h^i(D) = 0$ for any $i \geq 2$. Thus by the Riemann-Roch theorem $h^0(D) - h^1(D) = \chi(D)$, where the right hand side depends only on the numerical properties of D. We say that D is *non-special* if

$$h^0(D)\ h^1(D) = 0$$

and *special* otherwise. If D is an effective class, i.e. $h^0(D) > 0$, then it is non-special if and only if $h^0(D) = \chi(D)$. Thus the *expected dimension* of D is $\max\{\chi(D), 0\}$. The aim of this note is to discuss two conjectures about special classes in dimension 2 and 3.

In Section 1 we introduce a quadratic form on $\mathrm{Pic}(X)$ which will be useful to describe the action of some birational automorphisms of X on $\mathrm{Pic}(X)$. The study of these maps, called small modifications, will be done in Section 2, while Section 3 will be devoted to the notions of pre-standard and standard forms for a class D. In Section 4 we introduce (-1)-classes and we study their properties with respect to the quadratic form. Section 5 contains the proof of the equivalence of two conjectures about special classes in dimension 2, one of these conjectures is the well-known S.H.G.H. conjecture, while the other is formulated in terms of standard classes.

Theorem. *Let X be the blow-up of $\mathbb{P}^2$ at a finite number of points in very general position. The following two statements are equivalent:*

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(1) *an effective class D is special if and only if there exists a (-1)-curve E such that $D \cdot E \leq -2$;*
(2) *an effective class in standard form is non-special.*

Section 6 deals with a conjecture about special classes in dimension 3. Here we recall the following (the complete formula for $h^0(D)$ is given in Conjecture 6.3).

Conjecture. *Let X be the blow-up of $\mathbb{P}^3$ at a finite number of points in very general position and let $D := dH - \sum_i m_i E_i$ be in standard form.*

(1) *If $q(D) \leq 0$, then $h^0(D) = h^0(D - Q)$.*
(2) *If $q(D) > 0$ then D is special if and only if $d < m_1 + m_2 - 1$.*

Finally Section 7 contains examples of calculation of $h^0(D)$ for some D. Several Maple programs will help the reader to concretely use the described algorithms for the calculation of $h^0(D)$ according to the proposed conjectures.

1. Basic setup

Let us first recall some definitions and fix some notations.

1.1. **Points in very general position.** Let $p_1, \dots, p_r$ be distinct points of $\mathbb{P}^n$ and let $m \in \mathbb{N}^r$. Consider the Hilbert scheme $(\mathbb{P}^n)^{[r]}$ parametrizing r-tuples of points in $\mathbb{P}^n$ and let $\mathcal{P} \in (\mathbb{P}^n)^{[r]}$ be the point corresponding to the p_i's. Denote by $\mathcal{H}(d, m, \mathcal{P})$ the vector space of degree d homogeneous polynomials of $\mathbb{C}[x_0, \dots, x_n]$ with multiplicity at least m_i at each p_i. Observe that $\dim \mathcal{H}(d, m, \mathcal{P})$ depends on $\mathcal{P}$ and that there is an open Zariski subset $\mathcal{U}(d, m) \subseteq (\mathbb{P}^n)^{[r]}$ where this dimension attains its minimal value. Let us denote by

$$\mathcal{U} := \bigcap_{(d,m) \in \mathbb{N}^{r+1}} \mathcal{U}(d, m). \tag{1.1}$$

Notation 1.1. From now on we will say that the points $p_1, \dots, p_n \in \mathbb{P}^n$ are in *very general position* if the corresponding $\mathcal{P}$ is in $\mathcal{U}$ (which is the complementary of a countable union of Zariski closed subspaces of the configuration space).
Moreover, given $p_1, \dots, p_r \in \mathbb{P}^n$ in very general position, we will denote by $\pi : X \to \mathbb{P}^n$ the blow-up of $\mathbb{P}^n$ at the p_i's, with exceptional divisors $E_1, \dots, E_r$ and by H the pull-back of a hyperplane of $\mathbb{P}^n$.

Proposition 1.2. *Let $D := dH - \sum_i m_i E_i$ with $d > 0$ and $m_i \geq 0$. Then $h^i(D) = 0$ for any $i > 1$.*

Proof. By abuse of notation denote by H a general representative of the class H. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(H) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(1) \longrightarrow 0.$$

Since X is rational, we have $h^i(\mathcal{O}_X) = 0$ for $i > 0$. Thus, taking cohomology, we get $h^i(H) = h^i(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = 0$ for $i > 0$. Now assume by induction that $h^i(d'H) = 0$ for $d' < d$ and $i > 0$. Tensoring the preceding sequence with $\mathcal{O}_X((d-1)H)$ and

taking cohomology one gets $h^i(dH) = h^i(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) = 0$ for $i > 0$ by [Har86, §III Thm. 5.1].

We now proceed by induction on $m := \sum_i m_i$. If $m = 0$ we have already proved the statement. Suppose it is true for $m' < m$ and let us prove it for m. We can assume $m_1 > 0$. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + E_1) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(m_1 - 1) \longrightarrow 0.$$

By [Har86, §III Thm. 5.1] we have $h^i(\mathcal{O}_{\mathbb{P}^{n-1}}(m_1 - 1)) = 0$ for $i > 0$. By induction hypothesis $h^i(D + E_1) = 0$ for $i > 1$. Thus we get the thesis. $\square$

Remark 1.3. If $D = dH - \sum m_i E_i \in \mathrm{Pic}(X)$ is effective and $m_i < 0$ for some i, then $E_i \subset \mathrm{Bs}\,|D|$. In fact, if we denote by e_i the class of a line in E_i, we have the following intersection products: $e_i E_j = -\delta_{i,j}$, $e_i H = 0$. Therefore $De_i = m_i < 0$, which implies that e_i is contained in $\mathrm{Bs}\,|D|$ and, since e_i spans the whole E_i, we get the claim.

1.2. **A quadratic form.** Consider the quadratic form on $\mathrm{Pic}(X)$ whose matrix with respect to the basis $H, E_1, \dots, E_r$ is diagonal with

$$H^2 = n - 1 \qquad E_1^2 = \cdots = E_r^2 = -1.$$

From now on $D_1 \cdot D_2$ will denote the value of the corresponding bilinear form defined by the quadratic form. Observe that the lattice $(\mathrm{Pic}(X), \cdot)$ has determinant $\pm(n-1)$, so that it is unimodular if and only if $n = 2$ in which case it coincides with the Picard lattice of X.

Definition 1.4. Let $R \in \mathrm{Pic}(X)$ with $R^2 = -2$. The *reflection* defined by R is the $\mathbb{Z}$-linear map:

$$\sigma_R : \mathrm{Pic}(X) \to \mathrm{Pic}(X) \qquad D \mapsto D + (D \cdot R)R.$$

Observe that σ_R is the reflection in $\mathrm{Pic}(X)$ with respect to the hyperplane orthogonal to R. We will denote by

$$F := H - E_1 - \cdots - E_{n+1} \qquad F_i := E_i - E_{i+1}, \text{ for } 1 \le i \le r - 1$$

Definition 1.5. We consider the following subgroups of $\mathbb{Z}$-linear isometries of $\mathrm{Pic}(X)$ defined by:

$$S(X) := \langle \sigma_i : 1 \le i \le r - 1 \rangle \qquad W(X) := \langle \sigma, S(X) \rangle,$$

where $\sigma_i := \sigma_{F_i}$ and $\sigma := \sigma_F$.

Remark 1.6. Observe that $S(X) \cong S_r$, the group of permutations on r elements, since σ_k corresponds to the transposition $(k, k+1)$. The group $W(X)$ is not necessarily finite. The class $K := \frac{1}{n-1} K_X$ is $W(X)$-invariant with $K \cdot F = K \cdot F_i = 0$ for any i. Thus, since the quadratic form has signature $(1, r-1)$, in case $K^2 = n + 3 + \frac{4}{n-1} - r > 0$, the restriction to $K^\perp$ is negative definite, and $W(X)$ is the Weyl group of the lattice $(K^\perp, \cdot)$. The following table describes $(K^\perp, \cdot)$ for all the values of n and r such that $K^2 > 0$ (see [DV81], [Dol83] and [Muk04] for a detailed discussion of these lattices). Observe that a set of simple roots for the lattice $(K^\perp, \cdot)$ is always given by F and the F_i's.

n	r	$K^\perp$
≥ 2	$\leq n+2$	A_r
≥ 2	$n+3$	D_{n+3}
4	8	E_8
3	7	E_7
2	6	E_6
2	7	E_7
2	8	E_8

2. Small modifications

The aim of this section is to relate the elements of $W(X)$ with some birational maps of X. In order to do that we first recall the following definition.

Definition 2.1. A *small modification* $\varphi : X_1 \dashrightarrow X_2$ is a birational map which is an isomorphism in codimension 1, i.e. there exist open subsets $U_i \subseteq X_i$, such that $\operatorname{codim}(X_i \setminus U_i) \geq 2$ and $\varphi_{|U_1} : U_1 \to U_2$ is an isomorphism.

Given a divisor $D \subseteq X_1$ one defines the isomomorphism:

$$\varphi_* : \operatorname{Div}(X_1) \to \operatorname{Div}(X_2) \qquad D \mapsto \overline{\varphi(D \cap U_1)}.$$

We will denote by the same symbol φ_* the induced isomomorphism $\operatorname{Pic}(X_1) \to \operatorname{Pic}(X_2)$.

An immediate consequence of Definition 2.1 is the following.

Proposition 2.2. *Let $\varphi : X_1 \dashrightarrow X_2$ be a small modification. Then $h^0(\varphi_*(D)) = h^0(D)$ for any $D \in \operatorname{Pic}(X_1)$.*

Let us go back now to the blow-up $\pi : X \to \mathbb{P}^n$ at r points $p_1, \dots, p_r$ in very general position, with $r \geq n+1$. We can suppose that the first $n+1$ points are the fundamental ones. Consider the small modification $\varphi : X \dashrightarrow X'$ induced by the birational map:

$$\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \qquad (x_0 : \cdots : x_n) \mapsto (x_0^{-1} : \cdots : x_n^{-1}),$$

where $\pi' : X' \to \mathbb{P}^n$ is the blow-up of $\mathbb{P}^n$ at $p'_i = p_i$, for $i \leq n+1$, and $p'_k = \phi(p_k)$ for $k > n+1$. Observe that by choosing $\{p_1, \dots, p_r\} \in \mathcal{U} \cap \phi(\mathcal{U})$ one has $\{p'_1, \dots, p'_r\} \in \mathcal{U} \cap \phi(\mathcal{U})$ (where $\mathcal{U}$ is as in (1.1)) so that the p'_i are still in very general position. Therefore, even if X and X' are not isomorphic, we can (and from now on we do) identify $\operatorname{Pic}(X')$ and $\operatorname{Pic}(X)$, so that φ_* can be considered as a $\mathbb{Z}$-linear map on $\operatorname{Pic}(X)$.

Proposition 2.3. *With the same notation as above we have $\varphi_* = \sigma$.*

Proof. Recall that $F = H - E_1 - \cdots - E_{n+1}$ and that $\sigma(D) = D + (D \cdot F)F$. Since the point p_i, with $i \leq n+1$, is mapped by φ to the hyperplane $x_{i+1} = 0$, then $\varphi_*(E_i) = E_i + F = \sigma(E_i)$. Moreover, from $p'_k = \varphi(p_k)$, we get $\varphi_*(E_k) = E_k = \sigma(E_k)$ for $k > n+1$. On the other hand, φ maps the hyperplane $x_0 = 0$ to p_1, since it is an involution. Thus $\varphi_*(F + E_1) = E_1 = \sigma(F + E_1)$. We conclude observing that $E_1, \dots, E_r$, $F + E_1$ form a basis of $\operatorname{Pic}(X)$. ∎

Proposition 2.4. *Let $D, D' \in \mathrm{Pic}(X)$; then for any $w \in W(X)$:*

1. $w(D) \cdot w(D') = D \cdot D'$;
2. $h^0(w(D)) = h^0(D)$, *moreover D is integral if and only if $w(D)$ is.*

Proof. The first statement follows from the fact that any $w \in W(X)$ is a composition of isometries of $(\mathrm{Pic}(X), \cdot)$. For the second statement observe that since the points $p_1, \dots, p_r$ of $\mathbb{P}^n$ are in very general position and σ_k exchanges p_k with p_{k+1}, then $h^0(\sigma_k(D)) = h^0(D)$ and moreover D is integral if and only if $\sigma_k(D)$ is integral. Observe that $h^0(D) = h^0(\varphi_*(D)) = h^0(\sigma(D))$, by Propositions 2.2 and 2.3. Moreover, since φ is an isomorphism on $U \subseteq X$, with $\mathrm{codim}(X \setminus U) \geq 2$, then $\overline{\varphi(D \cap U)}$ is integral if and only if D is integral. This completes the proof, by Proposition 2.3. □

3. Classes in standard form

In this section, given a class $D \in \mathrm{Pic}(X)$ we find a representative D' in the orbit $W(X) \cdot D$ which we will call in pre-standard form (see [LU06] and [Dum09]). We will see in the following sections that these objects play an important rule in the formulation of conjectures for special divisors in the blow up of $\mathbb{P}^2$ and $\mathbb{P}^3$.

Definition 3.1. A class $D := dH - \sum_i m_i E_i$ is in *pre-standard form* if one of the following equivalent conditions holds:

1. $D \cdot (H - (n-1)E_1) \geq 0$, $D \cdot F_i \geq 0$ and $D \cdot F \geq 0$, for any $i = 1, \dots, r-1$;
2. $d \geq m_1 \geq \cdots \geq m_r$ and $(n-1)d \geq m_1 + \cdots + m_{n+1}$.

If in addition $D \cdot E_r \geq 0$, or equivalently $m_r \geq 0$, then D is in *standard form.*

Proposition 3.2. *Let $D \in \mathrm{Pic}(X)$ be an effective class. Then there exists a $w \in W(X)$ such that $w(D)$ is in pre-standard form.*

Proof. Write $D := dH - \sum_i m_i E_i$ and observe that $d \geq m_1$ since $h^0(D) > 0$. We proceed by induction on $d \geq 0$. If $d = 0$, then $m_i \leq 0$ for any i, and applying an element of $S(X)$ we obtain a divisor D' in pre-standard form. Assume now that $d > 0$ and that the statement is true for $d' < d$. After applying an element of $S(X)$ we can assume that $m_1 \geq \cdots \geq m_r$. If $D \cdot F < 0$, then $\sigma(D) = d'H - \sum_i m_i' E_i$ with $d' = d + D \cdot F < d$. By induction hypothesis there exists a $w' \in W(X)$ such that $w'(\sigma(D))$ is pre-standard. By taking $w := w' \circ \sigma$ we get the thesis. □

3.1. **An algorithm for the pre-standard form.** The Maple program `std` is part of the package `StdClass` that can be freely downloaded (see [LU10]). Given a class $D := dH - \sum_i m_i E_i$, it returns its pre-standard form $D' = d'H - \sum_i m_i' E_i$. Here $n = \dim(X)$.

$$\begin{aligned} \text{INPUT} &= \; n, [d, m_1, \dots, m_r]. \\ \text{OUTPUT} &= \; [d', m_1', \dots, m_r']. \end{aligned}$$

Here is a Maple session.

```
> with(StdClass):
> std(3,[4,3,3,3,3]);
  [0, -1, -1, -1, -1]
```

4. (-1)-Classes

We are now going to introduce some other particular classes in $\mathrm{Pic}(X)$, i.e. the *(-1)-classes*, which turn out to be a generalization of (-1)-curves of $\mathbb{P}^2$. Next we analyze the relation between classes in standard form and (-1)-classes.

Definition 4.1. A (-1)-class E is an integral class with $h^0(E) > 0$ such that $E^2 = E \cdot K = -1$, where $K := \frac{1}{n-1} K_X$.

Observe that each E_i is a (-1)-class and that if $n = 2$, then (-1)-classes correspond to (-1)-curves.

Lemma 4.2. *Let E be a (-1)-class such that $E \cdot F_i \geq 0$ and $E \cdot E_i \geq 0$. Then $E \cdot F < 0$.*

Proof. Let $E := dH - \sum_i m_i E_i$. By hypothesis the multiplicities are in a decreasing order and $m_{n+1} \geq 0$. Therefore $E^2 + m_{n+1} E \cdot K < 0$, or equivalently

$$d((n-1)d - m_{n+1}(n+1)) - \sum_{i=1}^{n+1} m_i(m_i - m_{n+1}) < \sum_{i=n+2}^{r} m_i(m_i - m_{n+1}).$$

Since the right hand side of the inequality is non-positive, the left hand side is negative. Writing $(n-1)d = \sum_{i=1}^{n+1} m_i + E \cdot F$ and substituting one obtains

$$\sum_{i=1}^{n+1} (d - m_i)(m_i - m_{n+1}) + d(E \cdot F) < 0$$

which implies the thesis. □

Proposition 4.3. *A class E is a (-1)-class if and only if there exists a $w \in W(X)$ such that $E = w(E_1)$.*

Proof. If $w \in W(X)$, then $w(E_1)^2 = w(E_1) \cdot K = -1$, since w is an isometry and $w(K) = K$. Moreover $w(E_1)$ is integral, by Proposition 2.4, so that $w(E_1)$ is a (-1)-class.

Assume now that $E := dH - \sum_i m_i E_i$ is a (-1)-class. Modulo an element of $S(X)$ we can assume that the multiplicities are in a decreasing order, or equivalently $E \cdot F_i \geq 0$. Moreover, if $m_i < 0$, then $E_i \subseteq \mathrm{Bs}\,|E|$, by Remark 1.3, so that $E = E_i$ since E is integral. Thus $w(E) = E_1$, where $w \in W(X)$ is the transposition $(1, i)$. We now proceed by induction on $m_{n+1} \geq -1$. We have already proved the first step of the induction. Assume the property is true for any $m_{n+1} < m$ and let us prove it for $m_{n+1} = m \geq 0$. By Lemma 4.2 we have $E \cdot F \leq 0$. Thus $\sigma(E) = d'H - \sum_i m'_i E_i$, with $m'_{n+1} = m_{n+1} + E \cdot F < m$ and we conclude by induction. □

We are now ready for the main theorem of this section.

Theorem 4.4. *Let D be in standard form. Then $w(D) \cdot E \geq 0$ for any (-1)-class E and any $w \in W(X)$.*

Proof. Let $E = dH - \sum_i m_i E_i$. We first prove by induction on $d \geq 0$ that $D \cdot E \geq 0$. If $d = 0$, then $E = E_i$ for some i. Since D is in standard form, then $D \cdot E_i \geq 0$. If $d > 0$, let E' be the divisor obtained from E by reordering the multiplicities in decreasing order, then $D \cdot E' \leq D \cdot E$ and E' is a (-1)-divisor which satisfies $E' \cdot F_i \geq 0$. Thus we can assume that both $E \cdot F_i \geq 0$ and $E \cdot E_i \geq 0$ are satisfied. By Lemma 4.2 we have $E \cdot F < 0$. Hence

$$D \cdot (\sigma(E) - E) = D \cdot (E \cdot F)F \leq 0$$

since $D \cdot F \geq 0$. Let $\sigma(E) = d'H - \sum_i m'_i E_i$, with $d' = d + E \cdot F < d$. By induction hypothesis we have $D \cdot \sigma(E) \geq 0$ which implies $D \cdot E \geq 0$.
In order to conclude the proof, let $w' \in W(X)$ be the inverse of w. Then $w(D) \cdot E = D \cdot w'(E) \geq 0$ since $w'(E)$ is a (-1)-class by Proposition 4.3. □

The following corollary shows that some geometric properties of (-1)-curves on a rational surface generalize to (-1)-classes.

Corollary 4.5. *Let $D \in \operatorname{Pic}(X)$ be an effective class and let E, E' be (-1)-classes.*

1. *If $D \cdot E < 0$, then $E \subseteq \operatorname{Bs}|D|$.*
2. *If E, E' both have negative product with D, then $E \cdot E' = 0$.*

Proof. Assume that $D \cdot E < 0$ and let $w \in W(X)$ be such that $w(D)$ is in pre-standard form, by Proposition 3.2. Write $w(D) = M + \sum_i a_i E_i$, where M is in standard form and $a_i > 0$ for any i. Observe that $w(D) \cdot w(E) = D \cdot E < 0$ by Proposition 2.4, and $M \cdot w(E) \geq 0$ by Theorem 4.4. This implies that $w(E) \cdot \sum_i a_i E_i < 0$, and in particular $w(E) \cdot E_i < 0$ for some i. Since $w(E)$ is integral and $E_i \subseteq \operatorname{Bs}|w(E)|$, by Remark 1.3, then $w(E) = E_i$. Still, by Remark 1.3, we get $w(E) \subseteq \operatorname{Bs}|w(D)|$, which proves the first statement.

For the second statement observe that reasoning as above, we get $w(E) = E_j$, $w(E') = E_k$. Thus we conclude observing that $E \cdot E' = E_j \cdot E_k = 0$.

□

5. Classes on the blow-up of $\mathbb{P}^2$

The aim of this section is to prove the equivalence between two conjectures about special classes $D \in \operatorname{Pic}(X)$ on the blow-up X of $\mathbb{P}^2$ at points in very general position. We will provide a Maple program for calculating $h^0(D)$, based on the second conjecture. We recall that a class $D := dH - \sum_i m_i E_i$ of X is in standard form if

$$d \geq m_1 \geq \cdots \geq m_r \geq 0 \qquad d \geq m_1 + m_2 + m_3.$$

Theorem 5.1. *Let X be the blow-up of $\mathbb{P}^2$ at a finite number of points in very general position. The following two statements are equivalent:*

(1) *an effective class D is special if and only if there exists a (-1)-curve E such that $D \cdot E \leq -2$;*
(2) *an effective class in standard form is non-special.*

Proof. Let us first prove that (1) $\Rightarrow$ (2). If D is an effective class in standard form, then $D \cdot E \geq 0$ for any (-1)-curve E, by Theorem 4.4. Hence D is non-special.

We now prove (2) $\Rightarrow$ (1). Let D be an effective divisor such that $D \cdot E \geq -1$ for any (-1)-curve E. Observe that if $D \cdot E = -1$, then $h^0(D) = h^0(D - E)$ and $\chi(D) = \chi(D - E)$, by the Riemann-Roch theorem. Thus we can assume that $D \cdot E \geq 0$ for any (-1)-curve E. By Proposition 3.2 there is a $w \in W(X)$ such that $D' := w(D)$ is in pre-standard form. Write $D' = d'H - \sum_i m'_i E_i$ and observe that $m'_i \geq 0$ by our last assumption. Thus D' is standard and hence non-special. $\square$

We recall that statement (1) is known in literature as the *Segre, Harbourne, Gimigliano, Hirschowiz conjecture*, or simply *S.H.G.H. conjecture* (see [Gim87], [Har86], [Hir89] and [Seg62]), and it has been checked in a number of cases. The equivalence between the Segre conjecture and part (1) of Theorem 5.1 has been proved in [CM01].

5.1. **An algorithm for calculating $h^0(D)$.** The Maple program `dim2` (see [LU10]), given $D := dH - \sum_i m_i E_i$, returns $h^0(D)$, assuming one of the two statements of Theorem 5.1 to be true.

$$\begin{aligned} \text{INPUT} &= [d, m_1, \dots, m_r]. \\ \text{OUTPUT} &= h^0(D). \end{aligned}$$

Here is a Maple session.

```
> with(StdClass):
> dim2([96,34,34,34,34,34,34,34,34]);
  1
```

6. Classes on the blow-up of $\mathbb{P}^3$

The aim of this section is to state a conjecture about special classes $D \in \operatorname{Pic}(X)$ on the blow-up X of $\mathbb{P}^3$ at points in very general position. We will provide a Maple program for calculating $h^0(D)$, based on this conjecture. We recall that a class $D := dH - \sum_i m_i E_i$ of X is in standard form if

$$d \geq m_1 \geq \cdots \geq m_r \geq 0 \qquad 2d \geq m_1 + m_2 + m_3 + m_4.$$

Let Q be the strict transform of the quadric through the first 9 points, or equivalently its class is $2H - E_1 - \cdots - E_9$. In what follows we assume that D is in standard form. We wish to provide a criterion for deciding when $Q \subseteq \mathrm{Bs}(|D|)$.

Proposition 6.1. *The divisor $D_{|Q}$ has non-negative intersection with any (-1)-curve of Q.*

Proof. Let f_1, f_2 be the pull-back of the classes of two rulings of the quadric and let $e_1, \dots, e_9$ be the nine exceptional curves. These classes form a basis of $\mathrm{Pic}(Q)$. Observe that Q is the blow-up of $\mathbb{P}^2$ at 10 points, with basis of the Picard group given by:

$$f_1 + f_2 - e_1, \quad f_1 - e_1, \quad f_2 - e_1, \quad e_2, \dots, e_9.$$

Since $e_i = E_{i|Q}$ and $f_1 + f_2 = H_{|Q}$, then the class z of $D_{|Q}$ has degree and multiplicities with respect to this basis given by:

$$2d - m_1, \quad d - m_1, \quad d - m_1, \quad m_2, \dots, m_9.$$

If $m_4 \leq d - m_1 < m_3$, then either z or $\sigma(z)$ is in standard form after reordering the multiplicities in decreasing order. In the remaining cases z is in standard form after reordering the multiplicities in decreasing order. We conclude by Theorem 4.4.

□

Thus, assuming the S.H.G.H. conjecture to be true for 10 points of $\mathbb{P}^2$ in very general position, we deduce that $D_{|Q}$ is non-special, or equivalently that $h^0(D_{|Q}) \cdot h^1(D_{|Q}) = 0$. We define $q(D) := \chi(D_{|Q})$ and observe that

$$q(D) = (d+1)^2 - \frac{1}{2}\sum_{i=1}^{9} m_i(m_i + 1).$$

Proposition 6.2. *Assume that the S.H.G.H. conjecture is true for 10 points of $\mathbb{P}^2$ in very general position. If $q(D) \leq 0$, then $h^0(D) = h^0(D - Q)$.*

Proof. By Riemann-Roch, Serre's duality and the fact that $D_{|Q}$ is non-special, we deduce $h^0(D_{|Q}) = 0$, so that $h^0(D - Q) = h^0(D)$ which proves the thesis. □

The following conjecture has been formulated for the first time in [LU06].

Conjecture 6.3. *Let X be the blow-up of $\mathbb{P}^3$ at a finite number of points in very general position and let $D := dH - \sum_i m_i E_i$ be in standard form.*

(1) *If $q(D) \leq 0$, then $h^0(D) = h^0(D - Q)$.*
(2) *If $q(D) > 0$, then D is special if and only if $d < m_1 + m_2 - 1$ and*

$$h^0(D) = \binom{d+3}{3} - \sum_{i=1}^{r} \binom{m_i + 2}{3} + \sum_{m_i + m_j > d+1} \binom{m_i + m_j - d + 1}{3}.$$

Conjecture 6.3 has been proved for $r \leq 8$ and any multiplicities (see [DVL07]), for $m_i \leq 4$ and any r (see [BB09] and [Dum08]). Observe that if $m_2+m_3 > d+1$, then $m_1+m_4 < d-1$ since D is in standard form. Hence in this case the sum on the right hand side of (2) is on the pairs (m_1, m_2), (m_1, m_3), (m_2, m_3). If $m_2+m_3 < d+1$,then the sum is over all the pairs (m_1, m_i), such that $m_1 + m_i > d + 1$.

6.1. **An algorithm based on Conjecture 6.3(1).** The Maple program `quad` (see [LU10]), given $D := dH - \sum_i m_i E_i$ returns a standard class $D' = d'H - \sum_i m'_i E_i$ with $h^0(D') = h^0(D)$ and $q(D') > 0$.

$$\begin{aligned} \text{INPUT} &= [d, m_1, \ldots, m_r]. \\ \text{OUTPUT} &= [d', m'_1, \ldots, m'_r]. \end{aligned}$$

Here is a Maple session.

```
> with(StdClass):
> quad([19,9,9,9,9,9,9,9,9,9]);
  [15, 7, 7, 7, 7, 7, 7, 7, 7, 7]
```

6.2. **An algorithm for calculating $h^0(D)$.** The Maple program `dim3` (see [LU10]), given $D := dH - \sum_i m_i E_i$, returns $h^0(D)$ according to Conjecture 6.3. It makes use of the function `quad`.

$$\begin{aligned} \text{INPUT} &= [d, m_1, \ldots, m_r]. \\ \text{OUTPUT} &= h^0(D). \end{aligned}$$

Here is a Maple session.

```
> with(StdClass):
> dim3([19,9,9,9,9,9,9,9,9,9]);
  60
```

7. EXAMPLES

In this section we consider several examples of effective classes $D \in \text{Pic}(X)$, where X is the blow-up of $\mathbb{P}^n$ at points in very general position. Denote by

$$L_3(d; m_1^{a_1}, \ldots, m_s^{a_s})$$

the class of the strict transform of a hypersurface of degree d through a_i points of multiplicity $\geq m_i$, for $i = 1, \ldots, s$.

7.1. **Pre-standard form.** Consider the class $D := L_n(n+1; n^{n+1})$. Then $\sigma(D) = L_n(0; (-1)^{n+1})$ is in pre-standard form. This proves that D is sum of (-1)-classes. Here we run a Maple test when $n = 5$.

```
> Std(5,[6,[5,6]]);
  [0, [-1, 6]]
```

7.2. **Dimension 2.** According to the S.H.G.H. conjecture or its equivalent statement given in Theorem 5.1(2), the class $L_2(d; m^r)$ is non-special if $d \geq 3m$. If $d < 3m$ and $r > 8$, then it is non-effective since its pre-standard form has negative degree. If $r \leq 8$, then we can have a special class, like for example $D := L_2(96; 34^8)$. We expect D to be non-effective but we have

```
> Dim2([96,[34,8]]);
  1
```

As shown by the following

```
> Std(2,[96,[34,8]]);
  [0, [-2, 8]]
```

we have that $w(D) = \sum_i 2E_i$ so that D is a sum of (-1)-classes as well.

7.3. **Dimension 3.** The class $L_3(d; m^r)$ is in standard form if $d \geq 2m \geq 0$. In this case, according to Conjecture 6.3, it is non-special if $(d+1)^2 - \frac{9}{2}m(m+1) > 0$ or equivalently if

$$d > -1 + \frac{3\sqrt{2m^2+2m}}{2}.$$

An example of a class D in standard form with $q(D) \leq 0$ is $L_3(2m+1; m^9)$, with $m \geq 8$. In this case $h^0(D) = 60$ does not depend on m, even if the expected dimension does.

```
> Quad([19,[9,9]]);
  [15, [7, 9]]
```

The expected dimension of this class is 55, while we have:

```
> Dim3([19,[9,9]]);
  60
```

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